

On Jump-Diffusive Driving Noise Sources Some Explicit Results and Applications

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Abstract

We study some linear and nonlinear shot noise models where the jumps are drawn from a compound Poisson process with jump sizes following an Erlang- m distribution. We show that the associated Master equation can be written as a spatial m^{th} order partial differential equation without integral term. This differential form is valid for state-dependent Poisson rates and we use it to characterize, via a mean-field approach, the collective dynamics of a large population of pure jump processes interacting via their Poisson rates. We explicitly show that for an appropriate class of interactions, the speed of a tight collective traveling wave behavior can be triggered by the jump size parameter m . As a second application we consider an exceptional class of stochastic differential equations with nonlinear drift, Poisson shot noise and an additional White Gaussian Noise term, for which explicit solutions to the associated Master equation are derived.

Keywords. Markovian jump-diffusive process. Compound Poisson noise sources with Erlang jump distributions. Higher order partial differential equations. Lumpability of Markov processes. Mean-field approach to homogeneous multi-agents systems. Flocking behavior of multi-agents swarms.

Mathematics classification numbers.

60H10 *Stochastic ordinary differential equations*

82C31 *Stochastic methods (Fokker-Planck, Langevin, etc.)*

60K35 *Interacting random processes; statistical mechanics type models*

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1 Introduction

On the real line \mathbb{R} , we shall consider scalar time-dependent Markovian stochastic processes X_t , ($t \in \mathbb{R}^+$ is the time parameter) characterized by stochastic differential equations (SDE) of the form:

$$\begin{cases} dX_t = -f(X_t)dt + \sigma(X_t, t)dW_t + q_{X_t, t}, \\ X_0 = x_0, \end{cases} \quad (1)$$

where W_t is a standard Wiener process with diffusion coefficient $\sigma(x, t)$, $q_{X_t, t}$ stands for a compound Poisson process (CPP) with Poisson rate $\lambda(X_t, t)$ and jump sizes drawn from a given probability density $\varphi(x)$ and where the drift $-f(x)$ reflects the deterministic behavior of the system. If necessary (i.e., if σ is space dependent), we will interpret (1) in the Itô sense. Accordingly, the Master equation governing the evolution of the conditional probability density function (pdf) $P(x, t|x_0, 0) = \text{Prob}\{X(t) \in [x, x+dx] | x_0, 0\}$ reads [1]:

$$\begin{aligned} \partial_t P(x, t|x_0, 0) = & \partial_x [f(x)P(x, t|x_0, 0)] + \frac{1}{2}\partial_{xx} [\sigma^2(x, t)P(x, t|x_0, 0)] \\ & - \lambda(x, t)P(x, t|x_0, 0) + \int_{-\infty}^{\infty} \varphi(x-z)\lambda(z, t)P(z, t|x_0, 0)dz. \end{aligned} \quad (2)$$

Note that for $\lambda(x, t) \equiv 0$, the solution to Eq.(1) is a diffusion process with continuous trajectories. In the generic case where the Poisson rates are strictly positive, these trajectories show jumps and hence are discontinuous.

Due to its extremely wide range of potential applications, Eq.(1) together with Eq.(2) deserved a long and still growing list of research records. In the last decade, quite a few new contributions became available (a non exhaustive list is [2, 3, 4, 5, 6, 7, 8]). The goals were either to write classes of explicit expressions for means, variances, Laplace transforms or even for $P(x, t|x_0, 0)$ or to express conditions ensuring the existence of finite time-invariant (i.e. stationary) probability measures. Our goal here is to add some new information to this general effort by:

- a) Deriving a new higher order partial differential equation – equivalent to (2) – valid when the jumps of the CPP are drawn from an Erlang- m probability law:

$$\varphi(x) = \mathcal{E}(m, \gamma; x) := \frac{\gamma^m x^{m-1} e^{-\gamma x}}{\Gamma(m)} \chi_{x \geq 0}, \quad m = 1, 2, \dots, \quad (3)$$

with rate parameter $\gamma > 0$ and where $\chi_{x \geq 0}$ is the indicator function of the event $\{x \geq 0\}$.

- b) Constructing a new soluble class of multi-agents dynamics in which agents with pure jumps (i.e. $\sigma(x, t) \equiv 0$) interact via their inhomogeneous Poisson rates $\lambda(x, t)$ and where the jumps are drawn from $\varphi(x)$ taken as an Erlang-2 distribution.
- c) Solving explicitly Eq.(2) when $f(x) = \beta \tanh(\beta x)$, $\sigma = 1$, and the jump sizes are symmetric: $\varphi(x) = \varphi(-x)$.

Pure jump processes with Erlangian jump sizes

Consider the dynamics in (1) with inhomogeneous Poisson rates $\lambda(x, t)$ and Erlangian jumps distribution with parameter m as defined in Eq.(3). In this case, the governing Master equation for $P_m(x, t) = P_m(x, t|x_0, 0)$ reads:

$$\begin{aligned} \partial_t(P_m(x, t)) - \partial_x[f(x)P_m(x, t)] - \frac{1}{2}\partial_{xx}[\sigma^2(x, t)P_m(x, t)] = \\ -\lambda(x, t)P_m(x, t) + \int_{-\infty}^x \frac{\gamma^m(x-z)^{m-1}e^{-\gamma(x-z)}}{\Gamma(m)}\lambda(z, t)P_m(z, t)dz. \end{aligned} \quad (4)$$

Proposition 1

For sufficiently smooth deterministic drift $f(x)$, Poisson rates $\lambda(x, t)$ and diffusion coefficient $\sigma^2(x, t)$ (all at least m times differentiable with respect to x), the integral form of the Master equation (4) can be rewritten as the m^{th} -order spatial differential equation¹:

$$[\partial_x + \gamma]^m \left(\partial_t P_m - \partial_x [f \cdot P_m] - \frac{1}{2}\partial_{xx} [\sigma^2 P_m] \right) = [\gamma^m - [\partial_x + \gamma]^m] (\lambda \cdot P_m) \quad (5)$$

Moreover for $\lambda(x, t) = \lambda(x)$ and $\sigma^2(x, t) = \sigma^2(x)$ a stationary distribution to (4) necessarily verifies:

$$-[\partial_x + \gamma]^m \left(\partial_x [f \cdot P_m] + \frac{1}{2}\partial_{xx} [\sigma^2 P_m] \right) = [\gamma^m - [\partial_x + \gamma]^m] (\lambda \cdot P_m). \quad (6)$$

The proof of Proposition 1 is given in Appendix A. For arbitrary drift terms and Poisson rates, explicit solutions to Eq.(5) or Eq.(6) are obviously difficult to derive. For convenience and later use, let us briefly list a few situations with $\sigma^2 = 0$ yielding tractable solutions.

¹We suppress the arguments x and t in $f(x)$, $\lambda(x, t)$, $\sigma(x, t)$ and $P_m(x, t|x_0, 0)$.

Stationary solutions

Here we suppose that the large t limit of P_m exists and we write $P_{s,m}(x) = \lim_{t \rightarrow \infty} P_m(x, t)$ for normalizable solutions to (6).

- For $m = 1$, $\lambda = \lambda(x)$, $\sigma(x, t) = 0$ and drift force $f(x)$, we have by (6):

$$[\partial_x + \gamma] (\partial_x [f(x)P_{s,1}]) = \partial_x (\lambda(x)P_{s,1}) \quad (7)$$

with the well known solution

$$P_{s,1}(x) = \frac{\mathcal{N}}{f(x)} e^{-\gamma x + \int^x \frac{\lambda(\xi)}{f(\xi)} d\xi}, \quad (8)$$

and where \mathcal{N} is the normalization factor. Clearly, the stationary regime $P_{s,1}(x)$ will actually be reached only when $\mathcal{N} < \infty$.

- For $m = 2$ we have

$$[\partial_x + \gamma]^2 \{ \partial_x (f(x)P_{s,2}) \} = [\partial_x^2 + 2\gamma\partial_x] (\lambda(x)P_{s,2}) \quad (9)$$

Introducing the notation $P_{s,2}(x) = e^{-\gamma x} Q(x) \stackrel{not.}{=} e^{-\gamma x} Q$, Eq.(9) takes, after elementary manipulations, the form:

$$f(x)[Q]_{xx} + 2[f(x)]_x [Q]_x + [f(x)]_{xx} Q = [\lambda(x)Q]_x + \lambda(x)\gamma Q. \quad (10)$$

Eq.(10) cannot be solved for general drift $f(x)$ and Poisson rate $\lambda(x)$. However, in the linear (Ornstein-Uhlenbeck) case with $f(x) = \alpha x$ and for constant rate $\lambda(x) = \lambda$, Eq.(10) reduces to:

$$\alpha x [Q(x)]_{xx} + (2\alpha - \lambda) [Q(x)]_x - \lambda \gamma Q(x) = 0. \quad (11)$$

Invoking $[9]^2$, the normalized stationary density reads:

$$P_{s,2}(x) = \gamma e^{\frac{\lambda}{\alpha} - \gamma x} \left[\frac{\alpha \gamma x}{\lambda} \right]^{\frac{\frac{\lambda}{\alpha} - 1}{2}} \mathbb{I}_{\frac{\lambda}{\alpha} - 1} \left(2 \sqrt{\frac{\gamma \lambda}{\alpha}} x \right) \quad (12)$$

where $\mathbb{I}_\nu(x)$ stands for the modified Bessel function of the first kind. Let us emphasize that Eq.(12) was also obtained in [5] by using Laplace transformations.

- For general m , arbitrary drift $f(x)$ and constant Poisson rate λ , the resulting dynamics is known as the nonlinear shot noise process and has been discussed e.g. in [4]. In most cases, only the Laplace transform of

²See the entry 9.1.53 with $q = 1/2$, $p = (\lambda/\alpha - 1)$, $p^2 - \nu^2 q^2 = 0$ and imaginary λ and simplify once by z .

$P_{s,m}(x)$ (resp. $P_m(x, t)$) can be given explicitly and, provided $P_{s,m}(x)$ exists, the j^{th} -order cumulant $\kappa_{s,m}^{(j)}$ of $P_{s,m}$ can be calculated using the relations:

$$\begin{cases} \kappa_{s,m}^{(j)} = \int_0^\infty x^j \lambda \frac{\Gamma(m, \gamma; x)}{f(x)} dx, & j = 1, 2, \dots \\ \Gamma(m, \gamma; x) := \int_x^\infty \mathcal{E}(m, \gamma; \xi) d\xi, \end{cases} \quad (13)$$

where $\Gamma(m, \gamma; x)$ is the incomplete gamma function [4].

Time dependent solutions

Time dependent solutions to (5) are available only for a restricted choice of drift terms f and Poisson rates λ . Explicit transient dynamics can be derived for constant drift $f(x) = k$, linear drift $f(x) = \alpha x$ and – rather remarkably – a non-linear interpolation between the two situations (discussed in section 3). The case of constant drift has been discussed in detail in [10]. The case for linear drift $f(x) = \alpha x$ and constant λ is presented in [5]. Let us recall that in this latter case, the Laplace transform $\hat{P}_m(u, t) := \int_{\mathbb{R}^+} e^{-ux} P_m(x, t | x_0, 0) dx$ reads as:

$$\hat{P}_m(u, t) = \exp \left\{ -x_0 u e^{-\alpha t} - \lambda \int_0^t \left(1 - \left[\frac{\gamma}{\gamma + u e^{-\alpha(t-x)}} \right]^m \right) dx \right\}. \quad (14)$$

which can be inverted for $m = 1$ yielding [5, 8]:

$$P_1(x, t) = \chi_z e^{-\lambda t} \left\{ \delta(z) + \frac{\lambda \gamma}{\alpha} (e^{\alpha t} - 1) e^{-\gamma z} {}_1F_1 \left(1 - \frac{\lambda}{\alpha}, 2; \gamma [1 - e^{\alpha t} z] \right) \right\}, \quad (15)$$

with $z = x - x_0 e^{-\alpha t}$, ${}_1F_1(a, b, z) := \sum_{n=0}^\infty \frac{\Gamma(a+n)}{\Gamma(a)} \frac{\Gamma(b)}{\Gamma(b+n)} \frac{z^n}{n!}$ and where χ_z is the indicator function. Note that when $(1 - \lambda/\alpha) = -n$ is integer valued, $P_1(x, t)$ is an elementary function. Indeed, in this case ${}_1F_1(-n; b; z)$ reduces to the n^{th} -order generalized Laguerre polynomial $L_n^{(1)}(z)$.

2 Multi-agents systems and flocking

As stated in the introduction, jump-diffusive noise sources do have a wide range of applications. The number of potential applications is naturally enlarged if we consider λ and/or σ as space dependent. Space correlations in the noise sources typically occur in the mean-field description of interacting particle systems and multi-agents modeling. We have in mind applications, where simple mutual interactions between agents (resp. particles) give rise to mean-field dynamics for the barycenter of the spatially distributed agents. Recent contributions directly relevant for our context here are the results derived by M. Balázs et al. [7] and the applications in [11] and [12]. These

papers show that under adequate conditions, the stationary barycentric dynamics of multi-agents systems develop traveling wave solutions. Generalizing on these results, we consider the case $f(x) = 0$, and $\sigma = 0$ for $m = 1$ and $m = 2$. According to Eq.(5), one immediately has:

$$[\partial_x + \gamma] (\partial_t P_1) = -\partial_x (\lambda \cdot P_1), \quad (m = 1) \quad (16)$$

$$[\partial_x + \gamma]^2 (\partial_t P_2) = -\partial_x [\partial_x + 2\gamma] (\lambda \cdot P_2), \quad (m = 2) \quad (17)$$

In the sequel, we shall assume that the shot noise rate $\lambda(x, t)$ is a strictly positive and monotone decreasing function in x , thereby potentially giving rise to traveling wave-type stationary distributions. For such a stationary propagating regime, we will have $\lim_{t \rightarrow \infty} \mathbb{E}_m \{X(t)\} = C_m t$, where C_m is a constant velocity and where $\mathbb{E}_m \{X(t)\} := \int_{\mathbb{R}} x P_m dx$. We therefore introduce the change of variable $\xi = x - C_m t$ and suppose that for large t , the jump rate is of the form:

$$\lambda(x, t) = \lambda(x - \mathbb{E}_m \{X(t)\}) = \lambda(\xi) \geq 0. \quad (18)$$

Under these assumptions, the equations in (16) can be rewritten as ODE's in $\xi \in \mathbb{R}$.

- For $m = 1$, we have:

$$-C_1(\gamma + \partial_\xi) \partial_\xi P_1(\xi) = -\partial_\xi \{[\lambda(\xi) P_1(\xi)]\}, \quad (m = 1) \quad (19)$$

admitting the traveling wave solution $P_1(\xi) = \mathcal{N} e^{-\gamma\xi + \int^\xi \frac{\lambda(z) dz}{C_1}}$ with \mathcal{N} being the normalization constant which must be self-consistently determined under the constraint $\int_{\mathbb{R}} \xi \cdot P_1(\xi) d\xi = 0$.

- For $m = 2$, after one immediate integration with respect to ξ , we have:

$$-C_2(\gamma + \partial_\xi)^2 P_2(\xi) = -\{2\gamma + \partial_\xi\} [\lambda(\xi) P_2(\xi)], \quad (m = 2) \quad (20)$$

which, if we introduce the auxiliary function $\Psi(\xi)$ defined through:

$$P_2(\xi) = \exp \left\{ -\gamma\xi + \int^\xi \frac{\lambda(z)}{2C_2} dz \right\} \Psi(\xi), \quad (21)$$

reduces to

$$\partial_{\xi\xi} \Psi(\xi) + \left[-\frac{\partial_\xi \lambda(\xi)}{2C_2} - \frac{\lambda^2(\xi)}{4C_2^2} - \frac{\gamma\lambda(\xi)}{C_2} \right] \Psi(\xi) = 0. \quad (22)$$

We observe that for arbitrary $\lambda(\xi)$, Eq.(22) exhibits the form of a stationary Schrödinger equation which, in general, cannot be solved in compact form. Looking for compact solutions to Eq.(22), the term in brackets can be related to analytically tractable potentials in quantum mechanics. To carry on the discussion for $m = 1$ and $m = 2$, we follow the lines exposed in [7] and focus on the special case which results, when the jump rates are of the form $\lambda(\xi) = e^{-\beta\xi}$, with $\beta > 0$.

Jump rate governed by $\lambda(\xi) = e^{-\beta\xi}$.

- The case $m = 1$ has been worked out in the mean-field context of interacting particle systems by Balazs et al. in [7], (see the Corollary 3.2). We find that $P_1(\xi)$ is a Gumbel-type distribution:

$$P_1(\xi) = \mathcal{N}(\beta, \gamma, C_1) e^{-\gamma\xi - \frac{1}{\beta C_1} e^{-\beta\xi}}, \quad (23)$$

with $\mathcal{N}(\beta, \gamma, C_1)$ being the normalization factor. The normalization \mathcal{N} and the resulting stationary velocity C_1 are explicitly found to be:

$$\mathcal{N}(\beta, \gamma, C_1) = \frac{\beta}{(\beta C_1)^{\frac{2}{\beta}} \Gamma(\gamma/\beta)} \quad (24)$$

$$C_1 = \frac{1}{\beta} e^{-\psi(\gamma/\beta)}, \text{ with } \psi(x) := \frac{d}{dx} \ln [\Gamma(x)] \quad (25)$$

ensuring $\int_{\mathbb{R}} P_1(\xi) d\xi = 1$ and $\int_{\mathbb{R}} \xi P_1(\xi) d\xi = 0$.

- For $m = 2$, Eq.(22) now reads:

$$\partial_{\xi\xi} \Psi(\xi) + \left[\frac{(\beta - 2\gamma)}{2C_2} e^{-\beta\xi} - \frac{1}{4C_2^2} e^{-2\beta\xi} \right] \Psi(\xi) = 0. \quad (26)$$

Observe that Eq.(26) corresponds to the stationary Schrödinger Eq. describing a quantum particle submitted to a Morse type potential for which explicit solutions are known. Using these results in the expression for $P_2(\xi)$ and imposing vanishing boundary conditions for large $|\xi|$ (see Appendix B for details), we find:

$$P_2(\xi) = \mathcal{N}(\beta, \gamma, C_2) e^{[\frac{\beta}{2} - \gamma]\xi - \frac{e^{-\beta\xi}}{2\beta C_2}} W_{\frac{\beta-2\gamma}{2\beta}, 0} \left(\frac{e^{-\beta\xi}}{\beta C_2} \right) \quad (27)$$

where $W_{\lambda, \mu}(z)$ is the Whittaker W function (see [13] 9.22) and $\mathcal{N}(\beta, \gamma, C_2)$ is the normalization factor. The normalization \mathcal{N} and the resulting stationary velocity C_2 are explicitly found to be:

$$\mathcal{N}(\beta, \gamma, C_2) = \frac{\beta}{(\beta C_2)^{\frac{\gamma}{\beta} - \frac{1}{2}}} \frac{\Gamma(2\gamma/\beta)}{\Gamma(\gamma/\beta)^2} \quad (28)$$

$$C_2 = \frac{1}{\beta} e^{\psi(2\gamma/\beta) - 2\psi(\gamma/\beta)}, \quad (29)$$

ensuring $\int_{\mathbb{R}} P_2(\xi) d\xi = 1$ and $\int_{\mathbb{R}} \xi P_2(\xi) d\xi = 0$. It is worthwhile noting that $C_2/C_1 = \exp(e^{\psi(2\gamma/\beta) - \psi(\gamma/\beta)}) > 2$, thus showing explicitly how the jump size parameter m influences the speed of the traveling wave solution $P_m(\xi)$.

3 Exactly soluble nonlinear mixed jump-diffusive processes

The mixed jump-diffusive processes defined by (1) do have the Markov property and are, under the assumption of sufficient symmetries, lumpable to

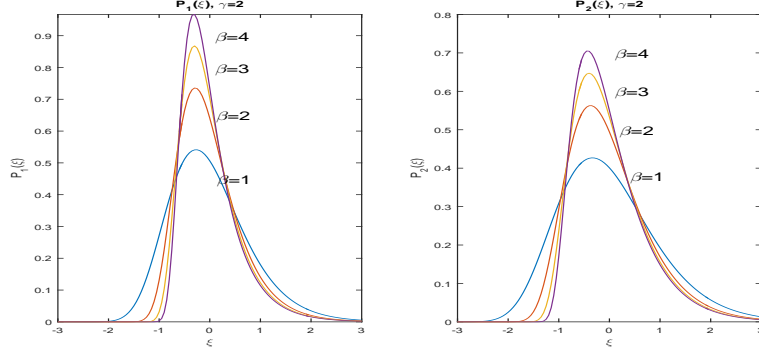


Figure 1: Exact normalized traveling probability waves $P_1(\xi)$ and $P_2(\xi)$ as given by Eqs.(23) respectively (27) for different values of β .

simpler processes [14]. In the realm of lumpable Markov diffusions, an outstanding role is played by Brownian motions with drift of the form $f(x) = \beta \tanh(\beta x)$ as they are, together with the class of Brownian motions with constant drift, the only ones having Brownian bridges as conditional laws [15]. This non linear and lumpable drift offers indeed the exceptional possibility to escape in a controlled and still analytical way from the Gaussian law (see e.g., [16, 17, 18]). This motivates us to consider the 1 dimensional dynamics given by:

$$\begin{cases} dX_t = \beta \tanh[\beta X_t] dt + dW_t + q_t, \\ X_0 = x_0, \end{cases} \quad (30)$$

where in this section q_t is a CPP with constant rate λ and jump sizes drawn from a symmetric probability law $\phi(x)$ (i.e., respecting $\phi(x) = \phi(-x)$ and $\int_{-\infty}^{\infty} \phi(x) dx = 1$). Here, we therefore can have positive and negative jumps. The Master equation Eq.(2) related to Eq.(30) reads:

$$\begin{aligned} \frac{\partial}{\partial t} Q(x, t|x_0) = & -\beta \frac{\partial}{\partial x} \{ \tanh(\beta x) Q(x, t|x_0) \} + \frac{1}{2} \partial_{xx} Q(x, t|x_0) \\ & - \lambda Q(x, t|x_0) + \lambda \int_{-\infty}^{\infty} Q(x-y, t|x_0) \phi(y) dy. \end{aligned} \quad (31)$$

By introducing the transformation $Q(x, t|x_0) = e^{-\frac{1}{2}\beta^2 t} \cosh(\beta x) R(x, t|x_0)$, it is immediate to verify that Eq.(31) takes the form:

$$\begin{aligned} \frac{\partial}{\partial t} R(x, t|x_0) = & \frac{1}{2} \partial_{xx} R(x, t|x_0) - \lambda R(x, t|x_0) \\ & + \frac{\lambda}{\cosh(\beta x)} \int_{-\infty}^{\infty} \cosh[\beta(x-y)] R(x-y, t|x_0) \phi(y) dy. \end{aligned} \quad (32)$$

The identity: $\cosh(a+b) = \cosh(a)\cosh(b) + \sinh(a)\sinh(b)$, enables to rewrite Eq.(32) as:

$$\begin{aligned} \frac{\partial}{\partial t} R(x, t|x_0) = & \frac{1}{2} \partial_{xx} R(x, t|x_0) + \lambda \int_{-\infty}^{\infty} R(x-y, t|x_0) \phi(y) \cosh(\beta y) dy \\ & - \lambda R(x, t|x_0) - \frac{1}{2} \lambda \tanh(\beta x) \int_{-\infty}^{\infty} \sinh(\beta y) R(x-y, t|x_0) \phi(y) dy. \end{aligned} \quad (33)$$

When the initial condition is taken $x_0 = 0$, symmetry of ϕ implies $Q(x, t|0) = Q(-x, t|0)$ and therefore also $R(x, t|0) = R(-x, t|0)$. Accordingly, when $x_0 = 0$, parity imposes the second integral in Eq.(33) vanishes. Hence Eq.(33) describes the evolution of the TPD $R(x, t|0)$ which characterizes a drift-free jump diffusion process $\tilde{X}(t)$, solution of

$$\frac{d}{dt}\tilde{X}(t) = dW_t + q_{\beta,t} \quad (34)$$

where now the Poisson noise $q_{\beta,t}$ – which is independent of the Wiener process W_t – is characterized by jumps drawn from the probability law $\phi_{\beta}(x) := \phi(x) \cosh(\beta x)$. Let us write $Q_{\beta}(x, t|0)$ for the TPD associated with the jump part in Eq.(34). Then we can write:

$$R(x, t|0) = \mathcal{N}(x, t|0) * Q_{\beta}(x, t|0) \quad (35)$$

where $*$ stands for the convolution in the space variable x and where $\mathcal{N}(x, t) := (\sqrt{2\pi t})^{-1} e^{-\frac{x^2}{2t}}$. Finally, for $x_0 = 0$, the TPD $Q(x, t|0)$ solving Eq.(31) reads:

$$\begin{cases} Q(x, t|0) = e^{-\frac{1}{2}\beta^2 t} \cosh(\beta x) \mathcal{N}(x, t|0) * Q_{\beta}(x, t|0) \\ \quad = \frac{1}{2} [\mathcal{N}^{(+\beta)}(x, t|0) + \mathcal{N}^{(-\beta)}(x, t|0)] * Q_{\beta}(x, t|0), \end{cases} \quad (36)$$

with $\mathcal{N}^{(\pm\beta)}(x, t) := (\sqrt{2\pi t})^{-1} e^{-\frac{(x-\beta t)^2}{2t}}$.

Illustration. The superposition of probability measures given by Eq.(36) can be used to derive explicitly new probability measures. For example, let us consider the case where in Eq.(31) we take:

$$\phi(x) = \frac{\gamma}{2} e^{-\gamma|x|}. \quad (37)$$

and for this choice, we consider the generalized Ornstein-Uhlenbeck dynamics Y_t characterized by:

$$dY_t = -\alpha Y_t dt + dX_t, \quad (38)$$

where in Eq.(38) the noise source dX_t is given by Eq.(30) with Eq.(37). The superposition given in Eq.(36) enables to write the TPD $P(y, t|y_0)$ characterizing the process Y_t as:

$$P(y, t|y_0) = \frac{1}{2} \left[P^{(+\beta)}(y, t|y_0) + P^{(-\beta)}(y, t|y_0) \right], \quad (39)$$

where $P^{(\pm\beta)}(y, t|y_0)$ are the TPD of the respective processes:

$$\begin{cases} dY_t^{(\beta)} = -\alpha Y_t dt + dX_{\beta,t} \\ dX_{\beta,t} = \pm\beta dt + dW_t + q_t \end{cases} \quad (40)$$

where q_t is the pure jump process with Poisson rate λ and jump size distribution $\frac{1}{2}\gamma e^{-\gamma|x|}$. Using now the results derived in [6, 2], we directly have³:

$$\begin{cases} \lim_{t \rightarrow \infty} P^{(\pm\beta)}(y, t|y_0) := P_s^{(\pm\beta)}(y) = \frac{2^\nu \gamma^{1-\nu} |y \pm \frac{\beta}{\alpha}|^{-\nu}}{\sqrt{\pi} \Gamma[\frac{1}{2}-\nu]} \mathbb{K}_\nu(\gamma|y \pm \frac{\beta}{\alpha}|), \\ \nu := \frac{1}{2} \left[1 - \frac{\lambda}{\alpha}\right], \end{cases} \quad (41)$$

where \mathbb{K}_ν is the modified Bessel function of the second kind. Consequently, the invariant measure $P_s(y)$ for the process Eq.(40) reads as:

$$\lim_{t \rightarrow \infty} P(y, t|y_0) = P_s(y) = \frac{1}{2} \left[P_s^{(-\beta)}(y) + P_s^{(+\beta)}(y) \right]. \quad (42)$$

Conclusion

Jump diffusions offer a rich class of noise sources and are widely used as modeling tools in various fields. As such, special interest lies in the explicit understanding of the effect of different jump distributions on the model dynamics. It is remarkable that in cases of space inhomogeneous shot noise with jump sizes following a gamma distribution with parameter (m, γ) , and space inhomogeneous jump frequency, $\lambda(\xi) = e^{-\beta\xi}$, a differential form of the Master-equation allows to quantitatively unveil the influence of the shape parameter m on the speed of stationary traveling wave solutions.

Appendix A

To the readers convenience, we give a detailed proof of proposition 1. We proceed by induction over $m \in \mathbb{N}$ (the Erlang parameter). We indeed show that (5) follows from (4) by applying the operator $\mathcal{O}_m := e^{-\gamma x} \partial_x^m e^{\gamma x}(\cdot)$ to (4), where ∂_x^m is the m -fold derivative with respect to x .

We start with the basic case by direct calculation and apply \mathcal{O}_m to (4) for $m = 1$ and use, for notational ease, $f(\cdot) = f$, $\lambda(\cdot, \cdot) = \lambda$, $\sigma(\cdot, \cdot) = \sigma$ and likewise $\partial_x(\cdot)$ or $(\cdot)_x$ for derivatives wrt x . We find:

$$\begin{aligned} e^{-\gamma x} \partial_x e^{\gamma x} \left(\partial_t P_1 - (f P_1)_x - \left(\frac{\sigma^2}{2} P_1 \right)_{x,x} \right) &= e^{-\gamma x} \partial_x e^{\gamma x} \left(-\lambda P_1 + \gamma \int_0^x e^{-\gamma(x-z)} \lambda P_1(z) dz \right) \\ [\gamma + \partial_x] \left(\partial_t P_1 - (f P_1)_x - \left(\frac{\sigma^2}{2} P_1 \right)_{x,x} \right) &= e^{-\gamma x} \left(-(\gamma e^{\gamma x} \lambda P_1 + e^{\gamma x} (\lambda P_1)_x) + \gamma e^{\gamma x} \lambda P_1 \right) \\ [\gamma + \partial_x] \left(\partial_t P_1 - (f P_1)_x - \left(\frac{\sigma^2}{2} P_1 \right)_{x,x} \right) &= -(\lambda P_1)_x \end{aligned}$$

which matches the proposition for $m = 1$.

For the induction step, we note I_m for the integral part of (4), i.e.:

$$I_m = \int_0^x \frac{\gamma^m (x-z)^{m-1} e^{-\gamma(x-z)}}{\Gamma(m)} \lambda(z, t) P_m(z, t|x_0, 0) dz$$

³See for instance Eq. (13) in [6].

and remark that $\partial_\gamma I_m = \frac{m}{\gamma} I_m - \frac{m}{\gamma} I_{m+1}$. Hence,

$$I_{m+1} = I_m - \frac{\gamma}{m} \partial_\gamma I_m \quad (43)$$

Let us apply $\mathcal{O}_{m+1} = e^{-\gamma x} \partial_x^{m+1} e^{\gamma x}(\cdot)$ to (4) for the case $m+1$. Using the Leibnitz formula for higher order derivatives of products⁴, the left hand side is immediately seen to be

$$[\gamma + \partial_x]^{m+1} (\partial_t P_{m+1} - (f P_{m+1})_x - (\frac{\sigma^2}{2} P_1)_{x,x}).$$

Apply the operator \mathcal{O}_{m+1} to the right hand side of (4) and use (43) to establish:

$$\begin{aligned} e^{-\gamma x} \partial_x^{m+1} e^{\gamma x} \left(-\lambda P_{m+1}(x, t) + I_{m+1} \right) = \\ e^{-\gamma x} \partial_x e^{\gamma x} \left\{ e^{-\gamma x} \partial_x^m e^{\gamma x} \right\} \left(-\lambda P_{m+1}(x, t) + I_m - \frac{\gamma}{m} \partial_\gamma I_m \right) \end{aligned}$$

Within the brackets we recognize \mathcal{O}_m which acts upon the left hand side of (4) for m and also on the extra term $(-\frac{\gamma}{m} I_m)$. Using the induction hypothesis, the right hand side reads:

$$e^{-\gamma x} \partial_x e^{\gamma x} \left([\gamma^m - [\partial_x + \gamma]^m] (\lambda \cdot P_{m+1}) - e^{-\gamma x} \partial_x^m e^{\gamma x} \frac{\gamma}{m} \partial_\gamma I_m \right)$$

A direct computation of the last term in the above parenthesis gives:

$$e^{-\gamma x} \partial_x^m e^{\gamma x} \frac{\gamma}{m} \partial_\gamma I_m = \gamma^m \lambda P_{m+1} - \int_0^x \gamma^{m+1} e^{-\gamma(x-z)} \lambda P_{m+1} dz.$$

We therefore are left to show that

$$\begin{aligned} e^{-\gamma x} \partial_x e^{\gamma x} \left([\gamma^m - [\partial_x + \gamma]^m] (\lambda \cdot P_{m+1}) - \gamma^m \lambda P_{m+1} + \int_0^x \gamma^{m+1} e^{-\gamma(x-z)} \lambda P_{m+1} dz \right) \\ \stackrel{!}{=} [\gamma^{m+1} - [\partial_x + \gamma]^{m+1}] (\lambda \cdot P_{m+1}) \quad (44) \end{aligned}$$

For the first term we get:

$$\begin{aligned} e^{-\gamma x} \partial_x e^{\gamma x} [\gamma^m - [\partial_x + \gamma]^m] (\lambda \cdot P_{m+1}) &= \gamma^m [\partial_x + \gamma] (\lambda \cdot P_{m+1}) - [\partial_x + \gamma]^{m+1} (\lambda \cdot P_{m+1}) \\ &= [\gamma^{m+1} - [\partial_x + \gamma]^{m+1}] (\lambda \cdot P_{m+1}) + \partial_x (\gamma^m \lambda P_{m+1}) \end{aligned}$$

For the middle term we find:

$$e^{-\gamma x} \partial_x e^{\gamma x} \left(-\gamma^m \lambda P_{m+1} \right) = -\gamma^{m+1} \lambda P_{m+1} - \partial_x (\gamma^m \lambda P_{m+1}) \quad (45)$$

finally the last term is:

$$e^{-\gamma x} \partial_x e^{\gamma x} \left(\int_0^x \gamma^{m+1} e^{-\gamma(x-z)} \lambda P_{m+1} dz \right) = \gamma^{m+1} \lambda P_{m+1} \quad (46)$$

Hence, adding (45)-(46) together, we have established (44) and therefore also proposition 1.

⁴The Leibnitz formula $\partial_x^m (f(x)g(x)) = \sum_{k=0}^m \binom{m}{k} f^{(k)} g^{(m-k)}$, with $f(x) = e^{\gamma x}$ takes the form $\partial_x^m (e^{\gamma x} g(x)) = e^{\gamma x} \sum_{k=0}^m \binom{m}{k} \gamma^k g^{(m-k)}$ and with the binomial formula we get $\partial_x^m (e^{\gamma x} g(x)) = e^{\gamma x} (\partial_x + \gamma)^m g(x)$

Appendix B

Our starting point is Eq.(26), (issued from Eq.(22) when $\lambda(\xi) = e^{-\beta\xi}$). First we introduce the change of variable

$$\begin{cases} Z = e^{-\beta\xi} \Rightarrow dZ = -\beta Z d\xi, \\ \partial_\xi(\cdot) \mapsto -\beta Z \partial_Z(\cdot), \\ \partial_{\xi\xi}(\cdot) \mapsto \beta^2 Z^2 \partial_{ZZ}(\cdot) + \beta^2 \partial_Z(\cdot). \end{cases} \quad (47)$$

In terms of the Z -variable, Eq.(26) takes the form:

$$\begin{cases} \beta^2 Z^2 \partial_{ZZ} \Psi(Z) + \beta^2 \partial_Z \Psi(Z) + [qZ - pZ^2] \Psi(Z) = 0, \\ q := \frac{(\beta-2\gamma)}{2C_2} \quad \text{and} \quad p := \frac{1}{4C_2^2}. \end{cases} \quad (48)$$

Or equivalently:

$$\partial_{ZZ} \Psi(Z) + \frac{1}{Z} \partial_Z \Psi(Z) + \left[\frac{q}{\beta^2 Z} - \frac{p}{\beta^2} \right] \Psi(Z) = 0. \quad (49)$$

Let us now write:

$$\Psi(Z) = Z^{-\frac{1}{2}} \varphi(Z). \quad (50)$$

Accordingly $\varphi(Z)$ obeys to the equation:

$$\partial_{ZZ} \varphi(Z) + \left[\frac{1}{4Z^2} + \frac{q}{\beta^2 Z} - \frac{p}{\beta^2} \right] \varphi(Z) = 0. \quad (51)$$

Now, let us introduce the rescaling:

$$\begin{cases} U = \omega Z, \\ \partial_Z(\cdot) \mapsto \omega \partial_U(\cdot) \quad \text{and} \quad \partial_{ZZ}(\cdot) \mapsto \omega^2 \partial_{UU}(\cdot) \end{cases} \quad (52)$$

Using Eq.(52) in Eq.(51), we obtain:

$$\partial_{UU} \varphi(U) + \left[\frac{1}{4U^2} + \frac{q}{\omega \beta^2 U} - \frac{p}{\omega^2 \beta^2} \right] \varphi(U) = 0 \quad (53)$$

Now, to match the standard Whittaker equation, (see entry 13.1.31 of [9]), we have to select:

$$\frac{p}{\omega^2 \beta^2} = \frac{1}{4} \Rightarrow \omega = \frac{1}{\beta C_2}. \quad (54)$$

So the general solution of Eq.(48) reads:

$$\Psi(\xi) = \sqrt{\beta C_2} e^{\frac{\beta\xi}{2}} \left\{ A M_{\frac{\beta-2\gamma}{2\beta}, 0} \left(\frac{e^{-\beta\xi}}{\beta C_2} \right) + B W_{\frac{\beta-2\gamma}{2\beta}, 0} \left(\frac{e^{-\beta\xi}}{\beta C_2} \right) \right\}, \quad (55)$$

where A and B are yet undetermined constants. By using Eq.(21), the probability density $P_2(\xi)$ reads:

$$P_2(\xi) = \mathcal{N} e^{-\gamma\xi - \frac{e^{-\beta\xi}}{2\beta C_2}} \Psi(\xi) = \mathcal{N} e^{[\frac{\beta}{2}-\gamma]\xi - \frac{e^{-\beta\xi}}{2\beta C_2}} \left\{ A M_{\frac{\beta-2\gamma}{2\beta},0} \left(\frac{e^{-\beta\xi}}{\beta C_2} \right) + B W_{\frac{\beta-2\gamma}{2\beta},0} \left(\frac{e^{-\beta\xi}}{\beta C_2} \right) \right\}. \quad (56)$$

where \mathcal{N} is the normalization factor. Let us now calculate the average of the positive definite function $\mathcal{G}(u)$ defined as:

$$\mathcal{G}(u) := \int_{-\infty}^{+\infty} e^{-u\xi} P_2(\xi) d\xi > 0. \quad (57)$$

and the normalization imposes that $\mathcal{G}(0) = 1$. Now, we introduce the new variable \mathcal{Z} defined as:

$$\mathcal{Z} := \frac{e^{-\beta\xi}}{\beta}, \quad (58)$$

In terms of this new variable, Eq.(56) now reads:

$$\mathcal{G}(u) = \mathcal{N} \int_0^\infty e^{\frac{\mathcal{Z}}{2C_2}} \mathcal{Z}^{\frac{\gamma+u}{\beta}-\frac{3}{2}} \left\{ A M_{\frac{1}{2}-\frac{\gamma}{\beta},0} \left(\frac{\mathcal{Z}}{C_2} \right) + B W_{\frac{1}{2}-\frac{\gamma}{\beta},0} \left(\frac{\mathcal{Z}}{C_2} \right) \right\} d\mathcal{Z}. \quad (59)$$

Now we use, the entries 7.622.8 and 7.622.11 from I. S. Gradshteyn to calculate \mathcal{I}_1 and \mathcal{I}_e with the choice of parameters $b = \frac{1}{C_2}$, $\mu = 0$, $\nu = \frac{\gamma+u}{\beta} - \frac{1}{2}$ and $\kappa = \frac{1}{2} - \frac{\gamma}{\beta}$ leading to :

$$\mathcal{G}(u) = A \left[\frac{\Gamma(1-\frac{2\gamma+u}{\beta})\Gamma(\frac{\gamma+u}{\beta})\Gamma(1-\frac{\gamma}{\beta})}{\Gamma(1-\frac{2\gamma}{\beta})\Gamma(\frac{\gamma}{\beta})\Gamma(1-\frac{\gamma+u}{\beta})} \right] (C_2)^{-\frac{u}{\beta}} + B \left[\frac{\Gamma(\frac{2\gamma}{\beta})\Gamma(\frac{\gamma+u}{\beta})^2}{\Gamma(\frac{\gamma}{\beta})^2\Gamma(\frac{2\gamma+u}{\beta})} \right] (C_2)^{\frac{u}{\beta}}. \quad (60)$$

As $\mathcal{G}(u) > 0$, the arguments of the Gamma functions have to be strictly positive for all values of γ and β . Hence, we are forced to impose $A = 0$ and hence $B = 1$. Let us now calculate the velocity C_2 , we end with

$$C_2 = 0 = -\frac{d}{du} \mathcal{G}(u) \big|_{u=0} = -\frac{d}{du} \left[e^{\frac{u}{\beta} \ln(C_2)} \varphi(u) \right] \big|_{u=0} \Rightarrow \quad (61)$$

$$C_2 = \frac{1}{\beta} e^{\frac{1}{\beta} [\Psi(\frac{2\gamma}{\beta}) - 2\Psi(\frac{\gamma}{\beta})]},$$

where $\Psi(x) := \frac{d}{dx} \ln[\Gamma(x)]$ is the digamma function

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